

# Mapping among manifolds 1

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## Abstract

In this paper we have build the modified Hamiltonian formalism for geometric objects like the Jacobi fields and metric tensors. In this approach Jacobi fields and metric tensors are mapped among manifolds. As an application, we have mapped a general  $n$ -dimensional Riemannian manifold to a  $n$ -dimensional maximally symmetric spacetime.

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# 1 Introduction

The Jacobi fields are very important to the Riemannian Geometry [1] and in the singularity theorems [2],[3]. These fields were used to study a free falling particle motion in a Schwarzschild spacetime [4], and a charged particle motion in Kaluza-Klein manifolds [5]. In this paper we present a map building method among manifolds. As an application, we have chosen, for their importance, a map between a n-dimensional general Riemannian manifold and a n-dimensional maximally symmetric spacetime.

This paper is organized as follows. In Sec. 2 we present some facts about the Jacobi fields. In Sec. 3 we build the Jacobi equation in a vielbein basis. In Sec. 4 we modify the Hamiltonian formalism to a new version, that we call *the modified Hamiltonian formalism*. In Sec. 5 we build a map among manifolds by the use of the modified Hamiltonian formalism for the Jacobi fields on a geodesic curve.. In Sec. 6 we apply the modified Hamiltonian formalism to the Jacobi fields on a non-geodesic curve. In Sec. 7 we apply the modified formalism to metric tensors. In Sec. 8 we summarize the main results of this work.

## 2 Jacobi Fields

In this section we briefly review the Jacobi fields and their respective differential equation for a Riemann manifold. Let us consider a differentiable manifold,  $\mathcal{M}$ , and two structures defined on  $\mathcal{M}$ , namely affine connection,  $\nabla$ , and Riemann tensor,  $K$ , related by the equation [2],[3]

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.1)$$

where  $X, Y$  and  $Z$  are vector fields in the tangent space. The torsion tensor can be defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (2.2)$$

where  $\nabla_X Y$  is the covariant derivative of the field  $Y$  along the  $X$  direction. We define the Lie derivative by

$$\mathcal{L}_X Y = [X, Y]. \quad (2.3)$$

For a torsion-free connection we can write

$$\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X. \quad (2.4)$$

Let us consider the tangent field  $V$ , the Jacobi field  $Z$  and the condition

$$\mathcal{L}_V Z = 0. \quad (2.5)$$

From (2.5), we can easily see that the equations which govern the Jacobi field can be written in the form

$$\nabla_V \nabla_V Z + K(Z, V)V - \nabla_Z \nabla_V V = 0. \quad (2.6)$$

In this section and in sections 3 and 5 we assume the simplest condition

$$\nabla_V V = 0. \quad (2.7)$$

In this case the Jacobi equations are reduced to a geodesic deviation, assuming a simpler form, and the Fermi derivative  $\frac{D_F Z}{\partial s}$  coincides with the usual covariant derivative

$$\frac{D_F Z}{\partial s} = \frac{DZ}{\partial s} = \nabla_V Z,$$

and

$$\nabla_V \nabla_V Z + K(Z, V)V = 0. \quad (2.8)$$

### 3 Jacobi Equation in a Vielbein Basis

We intend to make use of a vielbein basis related to a local metric  $G$ , as usual. The metric tensor  $G_{\Lambda\Pi}$  has signature  $(-, +, \dots, +)$ , with curved indices  $\Lambda, \Pi \in (0, 1, 2, 3, 4, 5, 6, \dots, n)$  and it is associated to a local coordinate basis. Let us consider the connection between the vielbein and the local metric tensor

$$G_{\Lambda\Pi} = E_{\Lambda}^{(\mathbf{A})} E_{\Pi}^{(\mathbf{B})} \eta_{(\mathbf{A})(\mathbf{B})}, \quad (3.1)$$

where  $\eta_{(\mathbf{A})(\mathbf{B})}$  and  $E_{\Lambda}^{(\mathbf{A})}$  are Lorentzian metric and vielbein components respectively. The flat indices  $(\mathbf{A}), (\mathbf{B}), \dots, (\mathbf{M}), (\mathbf{N}) \in (0, 1, 2, 3, 4, 5, 6, \dots, n)$ . In the vielbein basis we use the following Riemannian curvature

$$\begin{aligned} K_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} = & -\gamma_{(\mathbf{A})(\mathbf{B})(\mathbf{C}),(\mathbf{D})} + \gamma_{(\mathbf{A})(\mathbf{B})(\mathbf{D}),(\mathbf{C})} \\ & + \eta^{(\mathbf{M})(\mathbf{N})} [\gamma_{(\mathbf{B})(\mathbf{A})(\mathbf{M})} (\gamma_{(\mathbf{C})(\mathbf{N})(\mathbf{D})} - \gamma_{(\mathbf{D})(\mathbf{N})(\mathbf{C})}) \\ & + \gamma_{(\mathbf{M})(\mathbf{A})(\mathbf{C})} \gamma_{(\mathbf{B})(\mathbf{N})(\mathbf{D})} - \gamma_{(\mathbf{M})(\mathbf{A})(\mathbf{D})} \gamma_{(\mathbf{B})(\mathbf{N})(\mathbf{C})}], \end{aligned} \quad (3.2)$$

where the  $\gamma_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}$  are the Ricci rotation coefficients. We now consider a massive test particle. Since the particle has a non-vanishing rest mass, it is convenient to define the tangent vector  $V$  as a timelike one, so that  $G(V, V) = -1$ . Let us build a Fermi-Walker transport. In the Fermi-Walker transported particle frame the equation (2.8) is given by

$$\frac{d^2 Z_A}{d\tau^2} + K_{0A0C} Z_C = 0, \quad (3.3)$$

where  $\tau$  is, in general, an affine parameter, which in our case is the proper time of the particle and  $Z_A$  are the vielbein components of the space-like vector  $Z$ , with  $G(Z, V) = 0$ , and  $A \in (1, 2, 3, 4, 5, 6, \dots, n)$ .

## 4 Modified Hamiltonian Formalism

We begin this section with the following first order non-linear differential equation

$$\frac{dy}{d\tau} = y^2, \quad (4.1)$$

with the initial conditions in  $t = 0$ ,  $y = y_0$ ,  $\frac{dy}{d\tau} = y_0^2$ , and solution

$$y = \frac{y_0}{1 - y_0 t}. \quad (4.2)$$

We present the first order linear differential equation

$$\frac{dy}{d\tau} + p(t)y = g(t), \quad (4.3)$$

and impose that (4.2) is also solution for (4.3). In this case we have

$$\left(\frac{y_0}{1-y_0t}\right)^2 + \frac{y_0}{1-y_0t}p(t) = g(t). \quad (4.4)$$

There is a great set of functions  $p(t)$  and  $g(t)$  that satisfy (4.4). Unfortunately it is necessary to first solve (4.1), and sometimes a first order non-linear differential equation has an implicit solution. Considering we have a first or second order non-linear differential system, is it possible to build a solution method that transfers the non-linearity of the non-linear system to the coefficients of a linear system? The answer is positive when Hamilton equations can be put in a special form. In this section we will build this method and in the other ones we will make some applications. For this we will use the modified Hamiltonian formalism, which is reduced to the Hamiltonian formalism when the transformations are canonical or symplectic. It is well-known that in the Hamiltonian formalism the Hamilton equations and the Poisson brackets will be conserved only by a canonical or symplectic transformation. In [4] we have changed the non-relativistic time-dependent harmonic oscillator [6],[7] to a general relativistic approach. In the modified-Hamiltonian formalism only Hamilton equations will be conserved, in the sense that they will be transformed into other Hamilton equations by a non-canonical or non-symplectic transformation, and the Poisson brackets will not be invariant. We now build a modified Hamiltonian formalism. Consider a time-dependent Hamiltonian  $H(\tau)$  where  $\tau$  is an affine parameter, in this case, the proper-time of the particle. Let us define  $2n$  variables that will be called  $\xi^j$  with index  $j$  running from 1 to  $2n$  so that we have  $\xi^j \in (\xi^1, \dots, \xi^n, \xi^{n+1}, \dots, \xi^{2n}) = (q^1, \dots, q^n, p^1, \dots, p^n)$  where  $q^j$  and  $p^j$  are coordinates and momenta, respectively. We now define the Hamiltonian by

$$H(\tau) = \frac{1}{2}H_{ij}\xi^i\xi^j, \quad (4.5)$$

where  $H_{ij}$  is a symmetric matrix. We impose that the Hamiltonian obeys the Hamilton equation

$$\frac{d\xi^i}{d\tau} = J^{ik}\frac{\partial H}{\partial \xi^k}. \quad (4.6)$$

The equation (4.6) introduces the symplectic  $J$ , given by

$$\begin{pmatrix} O & I \\ -I & O \end{pmatrix} \quad (4.7)$$

where  $O$  and  $I$  are the  $n \times n$  zero and identity matrices, respectively. We now make a linear transformation from  $\xi^j$  to  $\eta^j$  given by

$$\eta^j = T^j_k \xi^k, \quad (4.8)$$

where  $T^j_k$  is a non-symplectic matrix, and the new Hamiltonian is given by

$$Q = \frac{1}{2} C_{ij} \eta^i \eta^j, \quad (4.9)$$

where  $C_{ij}$  is a symmetric matrix. The matrices  $H$ ,  $C$ , and  $T$  obey the following system

$$\frac{dT^i_j}{d\tau} + \frac{dt}{d\tau} T^i_k J^{kl} X_{lj} = J^{im} Y_{ml} T^j_k, \quad (4.10)$$

where  $2X_{lj} = \frac{\partial H_{ij}}{\partial \xi^l} \xi^i + 2H_{lj}$  and  $2Y_{ml} = \frac{\partial C_{il}}{\partial \eta^m} \eta^i + 2C_{ml}$ ,  $t$  and  $\tau$  are the proper-times of the particle in two different manifolds. We note that (4.10) is a first order linear differential equation system in  $T^i_k$ , and it is the response for what we looked for because the non-linearity in the Hamilton equations were transferred to their coefficients. Consider  $\frac{dt}{d\tau} X_{lj} = Z_{lj}$  and write (4.10) in the matrix form

$$\frac{dT}{d\tau} + TJZ = JYT, \quad (4.11)$$

where  $T$ ,  $Z$  and  $Y$  are  $2n \times 2n$  matrices as

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \quad (4.12)$$

with similar expressions for  $Z$  and  $Y$ . Let us write (4.11) as follows

$$\dot{T}_1 = Y_3 T_1 + Y_4 T_3 + T_2 Z_1 - T_1 Z_3, \quad (4.13)$$

$$\dot{T}_2 = Y_3 T_2 + Y_4 T_4 + T_2 Z_2 - T_1 Z_4, \quad (4.14)$$

$$\dot{T}_3 = -Y_1 T_1 - Y_2 T_3 + T_4 Z_1 - T_3 Z_3, \quad (4.15)$$

$$\dot{T}_4 = -Y_1 T_2 - Y_2 T_4 + T_4 Z_2 - T_3 Z_4. \quad (4.16)$$

Now consider

$$\dot{S}_1 = Y_3 S_1 + Y_4 S_3, \quad (4.17)$$

$$\dot{S}_2 = Y_3 S_2 + Y_4 S_4, \quad (4.18)$$

$$\dot{S}_3 = -Y_1 S_1 - Y_2 S_3, \quad (4.19)$$

$$\dot{S}_4 = -Y_1 S_2 - Y_2 S_4, \quad (4.20)$$

and

$$\dot{R}_1 = R_2 Z_1 - R_1 Z_3, \quad (4.21)$$

$$\dot{R}_2 = R_2 Z_2 - R_1 Z_4, \quad (4.22)$$

$$\dot{R}_3 = R_4 Z_1 - R_3 Z_3, \quad (4.23)$$

$$\dot{R}_4 = R_4 Z_2 - R_3 Z_4. \quad (4.24)$$

From the theory of first order differential equation systems [8], it is well-known that the system (4.17)-(4.24) has a solution in the region where  $Z_{lj}$  and  $Y_{ml}$  are continuous functions. In this case, the solution for (4.10) or (4.11) is given by

$$T_1 = (S_1 a + S_2 b) R_1 + (S_1 d + S_2 c) R_3, \quad (4.25)$$

$$T_2 = (S_1 a + S_2 b) R_2 + (S_1 d + S_2 c) R_4, \quad (4.26)$$

$$T_3 = (S_3 a + S_4 b) R_1 + (S_3 d + S_4 c) R_3, \quad (4.27)$$

$$T_4 = (S_3 a + S_4 b) R_2 + (S_3 d + S_4 c) R_4, \quad (4.28)$$

where a,b,c and d are constant  $n \times n$  matrices, and substituting (4.25)-(4.28) in (4.8) we will be completed the mapping among manifolds. In many situations where it is not possible or easy to put  $\frac{dt}{d\tau}$ ,  $X_{lj}$ ,  $Y_{lj}$  as explicit functions of one of the two parameters,  $t$  or  $\tau$ , we should expand them in series of  $\tau$ , for example,[8] so that with the modified Hamiltonian formalism we can map one differential equation system into another. In this paper we are interested in mapping among manifolds by the use of Hamiltonians for Jacobi fields and also for metric tensors. Therefore, if we associate  $H_{ij}$  and  $C_{lk}$  with two vielbein curvatures  $K_{0A0C}$  and  $R_{0A0C}$  respectively, or  $H_{ij}$  and  $C_{lk}$  with the metric tensors  $G_{ij}$  and  $\Omega_{ij}$ , also from two different manifolds, we will have built a local map among manifolds. It is important to note that the same particle has different proper-times in different manifolds, so that line elements are not preserved by local non-symplectic maps among manifolds, in opposition to general relativity where line elements are preserved by local coordinate transformations. The derivative  $\frac{dt}{d\tau}$  increases the difficulty in (4.10), so that we assume the condition  $\frac{dt}{d\tau} = 1$ . It implies in a decrease on

mapped regions. The local non-symplectic maps are well defined for equal proper-times and time intervals. In this paper, for the same particle in different manifolds with different proper-times, we only use the proper-time of one of the manifolds, so that (4.10) assume the following form

$$\frac{dT^i_j}{d\tau} + T^i_k J^{kl} X_{lj} = J^{im} Y_{ml} T^j_k. \quad (4.29)$$

As consequence (4.21)-(2.24) will be simplified. We end this section calling to mind that in the Hamiltonian formalism the Poisson bracket will be an invariant only by a canonical or symplectic transformation so that in this case (4.8) will be canonical or symplectic and it can be non-linear. In the modified Hamiltonian formalism only Hamilton equations will be conserved in the sense that they will be transformed into other Hamilton equations by a non-canonical or non-symplectic transformation (4.8), where (4.10) or (4.29) also will be obeyed, and the Poisson brackets will not be invariant.

## 5 Modified Formalism and Jacobi Fields

In this section we assume the condition

$$\nabla_V V = 0, \quad (5.1)$$

where  $G(V, V) = -1$ . We now identify the hamilton matrices with Jacobi fields as follow. Let us consider

$$H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j = \frac{1}{2} (\pi^t M^t \zeta + \zeta^t M \pi), \quad (5.2)$$

where  $\xi^j \in (\xi^1, \dots, \xi^n, \xi^{n+1}, \dots, \xi^{2n}) = (\zeta^1, \dots, \zeta^n, \pi^1, \dots, \pi^n)$  and  $Z_A = \zeta^j$  and  $\pi^j$  are coordinates and momenta, respectively, and  $H_{ij}$  is a real and symmetric matrix given by

$$\begin{pmatrix} O & M \\ M^t & O \end{pmatrix} \quad (5.3)$$

where  $O$  is the  $n \times n$  zero matrix,  $M^t$  is the  $n \times n$  transposed matrix of  $M$  with  $M_{AC} = V_{C;A}$ , and  $A, C \in (1, \dots, n)$ . Using the Hamiltonian (5.2) in the Hamilton equation, where  $Z_A = \zeta^j$ , we obtain (3.3). Let us consider

$$Q = \frac{1}{2} C_{ij} \eta^i \eta^j, \quad (5.4)$$



where

$$C_{lk} = \frac{1}{2}, \quad (5.5)$$

for  $l, k \in (n+1, \dots, 2n)$ . We use a vielbein basis related to a local metric  $G$ , as usual. The metric tensor  $G_{\Lambda\Pi}$  has signature  $(-, +, +, -, -, \dots, -, +, -)$ , with curved indices  $\Lambda, \Pi \in (0, 1, 2, 3, 4, 5, 6, \dots, n)$  and it is associated to a local coordinate basis. Let us consider the connection between the vielbein and the local metric tensor

$$G_{\Lambda\Pi} = E_{\Lambda}^{(\mathbf{A})} E_{\Pi}^{(\mathbf{B})} \eta_{(\mathbf{A})(\mathbf{B})}, \quad (5.6)$$

where  $\eta_{(\mathbf{A})(\mathbf{B})} = b_l \delta_{lk}$ , and  $E_{\Lambda}^{(\mathbf{A})}$  are a pseudo-euclidian metric and vielbein components respectively, and flat indices  $(\mathbf{A}), (\mathbf{B}), \mathbf{l}, \mathbf{k}, \dots \in (0, 1, 2, 3, \dots, n)$ . The curvature components in the Fermi-Walker transported particle frame are given by

$$C_{lk} = \frac{1}{2} K_l \delta_{lk}, \quad (5.7)$$

for  $\mathbf{l}, \mathbf{k} \in (1, 2, 3, 4, 5, 6, \dots, n)$ , where each  $K_l$  is a real constant, positive or negative,  $\delta_{lk}$  is the Kronecker's delta function. Using now the Hamiltonian (5.4) in the Hamilton equation, we obtain the following system

$$\frac{d^2 x_i}{d\tau^2} + K_i \delta_{ij} x_j = 0, \quad (5.8)$$

where  $x_i = \eta_i$  are the Jacobi fields in the manifold given by (5.5) and (5.7). It is a  $n$ -dimensional maximally symmetric space or a pseudo-sphere if  $K_l = K$ , where  $K$  is a positive constant. For a more general case, where each  $K_l$  is a real constant, positive or negative, we have built a local map between one general Riemannian manifold and another one that is compact in some directions and non-compact in other directions. For  $K_l$  as a real and positive constant, the system (5.8) also appears in the analysis of the Jacobi fields on a  $n$ -sphere [9], or geodesic motion on a  $n$ -dimensional maximally symmetric space-time [10], both embedded in a  $(n+1)$ -dimensional Euclidian manifold. In this case the curvature has the following expression

$$K_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} = K_{\Lambda\Pi\Omega\Theta} E_{(\mathbf{A})}^{\Lambda} E_{(\mathbf{B})}^{\Pi} E_{(\mathbf{C})}^{\Omega} E_{(\mathbf{D})}^{\Theta}, \quad (5.9)$$

where

$$K_{\Lambda\Pi\Omega\Theta} = K(G_{\Lambda\Theta} G_{\Pi\Omega} - G_{\Lambda\Omega} G_{\Pi\Theta}), \quad (5.10)$$

and in the Fermi-Walker transported particle frame (5.9) assume the simple form

$$K_{0A0C} = K\delta_{AC}. \quad (5.11)$$

We note that (5.11) is a special case of (5.7). Sometimes, as in [4], is convenient to assume the following Hamiltonian matrix elements

$$H_{ij} = K_{0A0C}, \quad (5.12)$$

where  $K_{0A0C}$  is the Riemannian curvature in a Fermi-Walker transported particle frame, and  $i, j, A, B \in (1, 2, 3, 4, 5, 6, \dots, n)$ . For  $i, j \in (n+1, \dots, 2n)$  we have

$$H_{ij} = \frac{1}{2}. \quad (5.13)$$

Explicitly,

$$H(\tau) = \frac{1}{2}H_{ij}\xi^i\xi^j = \frac{1}{2}(P_AP^A + Z^AK_{0A0C}Z^C). \quad (5.14)$$

Substituting (5.14) into Hamilton equation we will obtain (3.3). Hamiltonians of type (5.14) are not appropriate for Jacobi fields on a non-geodesic curve.

## 6 Jacobi Fields on a Non-Geodesic Curve

In this section we assume the condition

$$\nabla_V V \neq 0, \quad (6.1)$$

where  $G(V, V) = -1$ . In this case the Jacobi equation is not reduced to a geodesic deviation, and the Fermi derivative  $\frac{D_F Z}{\partial s}$  does not coincide with the usual covariant derivative

$$\frac{D_F Z}{\partial s} \neq \frac{DZ}{\partial s} = \nabla_V Z.$$

We intend to make use of a vielbein basis related to a local metric  $G$ , as in section 3. The metric tensor  $G_{\Lambda\Pi}$  has signature  $(-, +, \dots, +)$ , with curved indices  $\Lambda, \Pi \in (0, 1, 2, 3, 4, 5, 6, \dots, n)$  and it is associated to a local coordinate basis. Let us consider the connection between the vielbein and the local metric tensor

$$G_{\Lambda\Pi} = E_{\Lambda}^{(\mathbf{A})} E_{\Pi}^{(\mathbf{B})} \eta_{(\mathbf{A})(\mathbf{B})}. \quad (6.2)$$

Considering a Fermi-Walker transported particle frame and also the associated Hamiltonian given by the following function

$$H(\tau) = \frac{1}{2}H_{ij}\xi^i\xi^j = \frac{1}{2}(\pi^t M^t \zeta + \zeta^t M \pi), \quad (6.3)$$

where  $\xi^j \in (\xi^1, \dots, \xi^n, \xi^{n+1}, \dots, \xi^{2n}) = (\zeta^1, \dots, \zeta^n, \pi^1, \dots, \pi^n)$  and  $Z_A = \zeta^j$  and  $\pi^j$  are coordinates and momenta, respectively, and  $H_{ij}$  is a real and symmetric matrix given by

$$\begin{pmatrix} O & M \\ M^t & O \end{pmatrix} \quad (6.4)$$

where  $O$  is the  $n \times n$  zero matrix,  $M^t$  is the  $n \times n$  transposed matrix of  $M$  with  $M_{AC} = V_{C;A}$ , and  $A, C \in (1, \dots, n)$ . Using the Hamilton equation we have

$$\frac{d\zeta^A}{d\tau} = \frac{\partial H}{\partial \pi^A} = V_{A;C} \zeta^C, \quad (6.5)$$

and

$$\frac{d\pi^C}{d\tau} = -\frac{\partial H}{\partial \zeta^C}. \quad (6.6)$$

By the derivative of (6.5) we obtain the following result [2]

$$\frac{d^2 \zeta_A}{d\tau^2} + (R_{0A0C} - \dot{V}_{A;C} - \dot{V}_A \dot{V}^C) \zeta_C = 0, \quad (6.7)$$

where (6.7) is the Jacobi equation on a non-geodesic curve and  $\dot{V}_A = V_{A;C} V^C$ . Let us consider a Fermi-Walker transported particle frame in a new manifold. The associated Hamiltonian is given by the following function

$$Q(\tau) = \frac{1}{2}C_{ij}\eta^i\eta^j = \frac{1}{2}(\Pi^t N^t \chi + \chi^t N \Pi) \quad (6.8)$$

where  $g(U, U) = -1$ ,  $\eta^j \in (\eta^1, \dots, \eta^n, \eta^{n+1}, \dots, \eta^{2n}) = (\chi^1, \dots, \chi^n, \Pi^1, \dots, \Pi^n)$  and  $\chi^j$  and  $\Pi^j$  are coordinates and momenta, respectively, and  $C_{ij}$  is a real and symmetric matrix given by

$$\begin{pmatrix} O & N \\ N^t & O \end{pmatrix} \quad (6.9)$$

where  $O$  is the  $n \times n$  zero matrix,  $N^t$  is the transpose of the  $n \times n$  matrix  $N$  with  $N_{AC} = U_{C;A}$ , and  $A, C \in (1, \dots, n)$ . Using The Hamilton equation we have the following results

$$\frac{d\chi^A}{d\tau} = \frac{\partial Q}{\partial \Pi^A} = U_{A;C} \chi^C, \quad (6.10)$$

and

$$\frac{d\Pi^C}{d\tau} = -\frac{\partial Q}{\partial \chi^C}. \quad (6.11)$$

By the derivative of (6.10) we obtain

$$\frac{d^2 \chi_A}{d\tau^2} + (R_{0A0C} - \dot{U}_{A;C} - \dot{U}_A \dot{U}^C) \chi_C = 0, \quad (6.12)$$

where (6.12) is the Jacobi equation on a non-geodesic curve in the new manifold. Now we can build a map between the old and the new manifolds by the non-symplectic linear transformation (4.8). From the theory of first order differential equation systems [8], it is well-known that the system (4.10) or (4.29) has a solution in the region where the elements of the matrices  $X_{lj}$  and  $Y_{ml}$  are continuous functions. In other words, we have mapped (6.7) on (6.12).

## 7 Modified Formalism and Metric Tensor

In this section we identify the Hamilton matrices with the metric tensors of two different manifolds. Let us assume that

$$H_{ij} = G_{ij}, \quad (7.1)$$

where  $G_{ij}$  is a local metric tensor, and for the new Hamiltonian we assume the simple case given by

$$C_{lk} = \frac{1}{2} b(\tau)_l \delta_{lk}, \quad (7.2)$$

where  $\tau$  is an affine parameter, and  $C_{lk}$  is a metric tensor with  $i, j, l, k \in (1, 2, 3, 4, 5, 6, \dots, n)$ . For  $i, j, l, k \in (n+1, \dots, 2n)$  we have

$$H_{ij} = \frac{1}{2}, \quad (7.3)$$

and

$$C_{lk} = \frac{1}{2}. \quad (7.4)$$

Explicitly,

$$H(\tau) = \frac{1}{2}H_{ij}\xi^i\xi^j = \frac{1}{2}(P_iP^i + X^iG_{ij}X^j), \quad (7.5)$$

and

$$Q(\tau) = \frac{1}{2}C_{ij}\eta^i\eta^j = \frac{1}{2}(p_ip^i + x^lb(\tau)_l\delta_{lk}x^k), \quad (7.6)$$

where  $H_{ij} = H_{ji} = G_{ij} = G_{ji}$ , and  $C_{lk} = C_{kl} = \frac{1}{2}b(\tau)_l\delta_{lk}$ . As each  $b_l$  can be a positive or negative function for the same time interval, we have built a local map between one general (torsion-free connection or not) manifold and another one that is compact in some directions and non-compact in other directions. If  $b(\tau)_l$  are time-independent ( $b(\tau)_l = b_l$ ), then, in this case, the map is equivalent to (3.1). In other words, it is possible to obtain, with this map, the same results obtained with vielbein. However, vielbein is also very useful for Fermi transport which is defined in a torsion-free connection manifold only. It is possible we chose, as Hamiltonian metric functions, the usual symmetric forms as follow

$$H(\tau) = \frac{1}{2}H_{ij}\xi^i\xi^j = \frac{1}{2}X^iG_{ij}X^j, \quad (7.7)$$

and

$$Q(\tau) = \frac{1}{2}C_{ij}\eta^i\eta^j = \frac{1}{2}x^lg_{lk}x^k. \quad (7.8)$$

We could have substituted (7.5) and (7.6) by (7.7) and (7.8), with  $g_{lk} = b(\tau)_l\delta_{lk}$ . We note that functions (7.7) and (7.8) are not conventional Hamiltonians, but their associated system (4.17)-(4.24) will be simpler than the correspondent to (7.5) and (7.6). Now we consider an interesting Hamiltonian function presented in the last section

$$H(\tau) = \frac{1}{2}H_{ij}\xi^i\xi^j = \frac{1}{2}(p^tM^tx + x^tMp), \quad (7.9)$$

where  $\xi^j \in ((\xi^1, \dots, \xi^n, \xi^{n+1}, \dots, \xi^{2n}) = (x^1, \dots, x^n, p^1, \dots, p^n))$  and  $x^j$  and  $p^j$  are coordinates and momenta, respectively, and  $H_{ij}$  is a symmetric matrix given by

$$\begin{pmatrix} O & M \\ M^t & O \end{pmatrix} \quad (7.10)$$

where  $O$  is the  $n \times n$  zero matrix,  $M^t$  is the  $n \times n$  transposed matrix of  $M$  with  $M_{ij} = G_{ji}$ , and  $i, j \in (1, \dots, n)$ . Now we write another Hamilton

$$Q(\tau) = \frac{1}{2} C_{ij} \eta^i \eta^j = \frac{1}{2} (P^t N^t X + X^t N P) \quad (7.11)$$

where  $\eta^j \in (\eta^1, \dots, \eta^n, \eta^{n+1}, \dots, \eta^{2n}) = (X^1, \dots, X^n, P^1, \dots, P^n)$  and  $X^j$  and  $P^j$  are coordinates and momenta, respectively, and  $C_{ij}$  is a real and symmetric matrix given by

$$\begin{pmatrix} O & N \\ N^t & O \end{pmatrix} \quad (7.12)$$

where  $O$  is the  $n \times n$  zero matrix,  $N^t$  is the transpose of the  $n \times n$  matrix  $N$  with  $N_{lk} = g_{kl}$ , and  $l, k \in (1, \dots, n)$ . It is important to note that in this case  $G_{ij}$  and  $g_{lk}$  may not have defined symmetries, and using (4.8) and (4.10) or (4.29) we transform the Hamilton equation for (of)  $G_{ij}$  into that for (of)  $g_{lk}$ . The last map is very important because it connects different areas both in mathematics and physics.

## 8 Concluding Remarks

It is known that the system (5.8) of  $n$ -dimensional harmonic oscillators (for positive  $K_l$ ) is the Jacobi equation on a maximally symmetric spacetime [9], as well as it is the geodesic equation on a spacetime with constant curvature [10]. For different sets of signs in  $b_l$ , the system composed by (7.2) and (7.4) has as solutions different sets of pseudo-euclidian spaces, where each one of them has form-invariant metric by a subgroup of  $GL(n, \mathbb{R})$ . In Sections 5 and 6 we have presented two ways of building a map among manifolds by the use of the equations (4.6), (4.8) and (4.10). Physics of wormholes is a very important research area [11], [12], [13], [14]. Maps among manifolds can be thought as wormholes, and can be considered as a different approach for this. It is also possible to build a traversable wormhole by a map between two different regions in the same manifold. Solutions of (4.7) can be very difficult, but in cases where they are possible, maps among manifolds will be a powerful option to *conformal maps*, *Fermi transport*, and *vielbein formalism*. Traditionally, expressions like *Hamilton equations* are associated with some kind of invariance property, as in the specialized literature. Invariance is a fundamental property in many theories, as in general relativity. However, if

we want to build maps among manifolds, the modified Hamiltonian formalism can be useful.

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